Revision 1

Problem 1. Consider the following models:

$$y_t = \alpha_0 + \alpha_1 t + e_t \tag{1}$$

$$y_t = \rho y_{t-1} + v_t \tag{2}$$

where e_t and v_t are i.i.d.(0,1), and $\rho = 1$. A central banker considers the two models above to explain the Gross Domestic Product (GDP).

(a) What is the name of each model?

Trending model (or deterministic trend model) and random walk model respectively.

(b) Derive the unconditional mean and variance of y_t implied by each model. Is any of the two models covariance stationary or/and weak dependent?

For model (1):

$$E(y_t) = \alpha_0 + \alpha_1\, t \tag{3}$$

$$Var(y_t) = Var(e_t) = 1 \tag{4}$$

$$Cov(y_t, y_{t+k}) = E[(y_t - E(y_t))(y_{t+k} - E(y_{t+k}))] = E(e_t e_{t+k}) = 0 \text{ for } k \neq 0 \tag{5}$$

So y_t is not covariance-stationary because the mean depends on t (but y_t is stationary around its trend). But it is weak-dependent because e_t is i.i.d. (weak dependant) which gives us a covariance equal to zero.

For model (2) by recursive substitution we have (assume y_0 not random):

$$y_t = \rho^t y_0 + \sum_{i=0}^{t} \rho^i v_{t-i} = y_0 + \sum_{i=0}^{t} v_{t-i} \tag{6}$$

$$E(y_t) = y_0 \tag{7}$$

$$Var(y_t) = \sum_{i=0}^{t} Var(v_{t-i}) + 2 \sum_{i,j=0, i \neq j}^{t} Cov(v_{t-i} v_{t-j}) = t \tag{8}$$

$$Cov(y_t, y_{t+k}) = E[(y_t - E(y_t))(y_{t+k} - E(y_{t+k}))] = E((\sum_{i=0}^{t} v_{t-i}) \sum_{i=0}^{t+k} v_{t+k-i}) \tag{9}$$

$$= \sum_{i=0}^{t} E(v_{t-i}^2) = t, \tag{10}$$

for $k \neq 0$ because the expected value of cross-terms is zero from the assumption on the error term. Because the variance and covariance are a function of time t, y_t from model (2) is not covariance-stationary. Moreover it is not weak dependent because covariance is not converging to zero as $k \to \infty$.

(c) Briefly explain what is meant by covariance-stationarity and weak dependence.

See book or slides

(d) Predicting future GDP is of major importance in decision making regarding investment, spending and hiring (among other things). Hence we are interested in the $h-$step ahead forecast given the last observed information: $E(y_{t+h}|y_t)$. Derive $E(y_{t+h}|y_t)$ from model (1) and (2) assuming $\rho = 1$.

For model (1):

$$E(y_{t+h}|y_t) = E(\alpha_0 + \alpha_1(t+h) + e_{t+h}|y_t) = \alpha_0 + \alpha_1(t+h) \tag{11}$$

For model (2):

$$E(y_{t+h}|y_t) = E(y_{t+h-1} + v_{t+h}|y_t) = E(y_{t+h-2} + v_{t+h-1} + v_{t+h}|y_t) \tag{12}$$

$$= ... = E(y_t + v_t + \ldots + v_{t+h}|y_t) = y_t \tag{13}$$

For model (2) the $h-$step ahead forecast is the last available information contrary to model (1) where the $h-$ahead forecast is the a straight line (the deterministic trend).

(e) When $|\rho| < 1$, $E(y_{t+h}|y_t) = \rho^h y_t$. What happens with the $h-$step ahead forecast as $h \to \infty$ in model (2) for $|\rho| < 1$ and $\rho = 1$?

When $\rho = 1$, as $h \to \infty$, the forecast is always the last available information. If $|\rho| < 1$ we see that $E(y_{t+h}|y_t) = \rho^h y_t \to 0$ as $h \to \infty$ so the influence of the last available information (y_t) loses importance.

(f) y_t in model (1) has trending behaviour, while y_t in model (2) with $\rho = 1$ has highly persistent behaviour. Show that y_t described by the model:

$$y_t = \delta + y_{t-1} + u_t \tag{14}$$

is highly persistent and has a clear linear trend, where u_t is i.i.d.(0,1).

By recursive substitution we have:

$$y_t = \delta + y_{t-1} + u_t = \delta + \delta + y_{t-2} + u_{t-1} + u_t = \ldots = t\delta + y_0 + \sum_{i=0}^{t} v_{t-i} \tag{15}$$

where $t\delta$ clearly shows the linear trend and the part $\sum_{i=0}^{t} v_{t-i}$ corresponds to the highly persistent series.